SPAZIO BV

$$BV(a,b) = \{u \in L^1(a,b) : u' \in \mathcal{M}((a,b))\} / \sim \qquad \text{SP. VETTORIALE}$$

$u \sim v$ se $u(x) = v(x)$ $\tilde{\forall} x$

$$\|u\|_{BV} = \|u\|_{L^1} + \underset{\substack{\uparrow \\ \text{VARIAZIONE TOTALE}}}{\|u'\|_{\mathcal{M}}} \qquad \text{NORMA SU } BV$$

OSS: $W^{1,1} \subseteq BV$ $\quad u' \in L^1$, CIOÈ È ASS. CONT. RISP. A $\mathcal{L}$

$$\|u\|_{W^{1,1}} = \|u\|_{BV}$$

OSS (RADON-NIKODYM): $u \in BV \Rightarrow u' = f\,dx + D^s u \qquad f \in L^1(a,b)$

$D^s u \perp \mathcal{L}$ CIOÈ $\exists\, E \subseteq (a,b)$, $|E| = 0$,

$|D^s u|((a,b) \setminus E) = 0$

$D^s u$ parte singolare di u'

$$D^s u = \sum \alpha_j \delta_{x_j} + D^c u \qquad D^c u(\{x\}) = 0 \quad \forall x$$

parte di salto   parte cantoriana

PROP. BV è uno sp. di Banach.

DIM. u_n di C. in BV $\quad u_n \xrightarrow[n]{} u$ in $L^1 \quad u_n' \xrightarrow[n]{} \mu \in \mathcal{M}$

$\forall \varphi \in C^1_c(a,b)$: $\int_a^b u\varphi' = \lim_n -\int \varphi \, du_n' = -\int \varphi \, d\mu$ cioè $\mu = u'$ e $u_n \to u$ in BV.

DEF: $u_n \xrightarrow[BV]{} u$ se $u_n \to u$ in L^1 e $u_n' \xrightarrow[*]{} u'$ in $\mathcal{M}$

OSS: $u_n \xrightarrow[BV]{} u \iff u_n \to u$ in L^1 e $\|u_n'\|_{\mathcal{M}} \leq C$ e $\|u'\|_{\mathcal{M}} \leq \liminf_n \|u_n'\|_{\mathcal{M}}$

OSS: $u \in BV \Rightarrow \underset{x<y}{|u(x^+) - u(y^-)|} \leq \int_x^y |u'| \leq \|u'\|_{\mathcal{M}} \Longrightarrow \|u\|_{L^\infty} \leq \left| \fint_a^b u \right| + \|u'\|_{\mathcal{M}}$

TEO (COMPATTEZZA) $u_n \in BV$, $\|u_n\|_{BV} \leq C$

$\Rightarrow \exists\, u \in BV$ e $u_{n_k} \xrightarrow[BV]{} u$.

DIM. $\|u_n'\|_{\mathcal{M}} \leq C \Rightarrow \exists\, \mu$ e n_k T.C. $u_{n_k}' \xrightarrow[*]{} \mu$ in $\mathcal{M}$.

SIA $x_0 \in (a,b)$ T.C. $u_n'(\{x_0\}) = 0$ e $\mu(\{x_0\}) = 0$ $\forall n$

$$u_n(x) = \begin{cases} u_n(x_0) + \int_{x_0}^{x} du_n' & \tilde{\forall}\, x > x_0 \\ u_n(x_0) - \int_{x}^{x_0} du_n' & \tilde{\forall}\, x < x_0 \end{cases}$$

$\|u_n(x_0)\| \leq C' \Rightarrow$ posso supporre $u_{n_k}(x_0) \xrightarrow[k]{} C$

$$u(x) = \begin{cases} C + \int_{x_0}^{x} d\mu & x > x_0 \\ C - \int_{x_0}^{x} d\mu & x < x_0 \end{cases}$$

$u \in BV$ CON $u' = \mu$

$u_{n_k}(x) \xrightarrow[k]{} u(x)$ $\tilde{\forall}\, x$

$\Rightarrow u_{n_k} \to u$ IN $L^1 \Rightarrow u_{n_k} \xrightarrow[BV]{} u$.

ESISTENZA IN BV

$L(x,z): (a,b)\times\mathbb{R}^n \to \mathbb{R}$ s.c.i., $z \to L(x,z)$ CONVESSA $\forall x$

$$L^\infty(x,z) = \lim_{t\to+\infty} \frac{L(x, z_0 + tz)}{t} : (a,b)\times\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$$

↑

FUNZIONE RECESSIONE

OSS: $L^\infty(x,z)$ NON DIP. DA z_0, $z \to L^\infty(x,z)$ CONVESSA E POSITIV. 1-OMOGENEA

$$L^\infty(x,tz) = t\,L^\infty(x,z) \text{ per } t>0$$

ES: $L(z) = \sqrt{1+z^2}$ $\qquad L^\infty(z) = \lim_{t\to+\infty} \frac{\sqrt{1+t^2z^2}}{t} = |z|$

TEO

$L(x,z)$ s.c.i. , $z \to L(x,z)$ convessa $\forall x$ [SEMICONTINUITÀ]

$L(x,z) \geq \alpha |z| \quad \alpha > 0 \quad \forall x$ [COERCIVITÀ]

$\exists$ min in BV di $\mathcal{L}(u) = \int_a^b L(x,f)\,dx + \int_a^b L^\infty(x,g)\,d|D^s u|$

$u' = f\,dx + D^s u \qquad D^s u = g\,|D^s u| \quad$ (R.N.)

$g \in L^1((a,b), |D^s u|) \quad |g| = 1 \quad |D^s u|$-q.o.